VI. *A Letter from Mr.* John Robertſon *to the* Preſident, *containing an Explanation of the late Dr.* Halley's *Demonſtration of the Analogy of the* Logarithmic Tangents *to the* Meridian Line, *or Sum of the* Secants.

S I R,

Read Nov. 22. 1750.

MY Curioſity having lately led me to peruſe ſeveral Books on the Art of Navigation, I was ſomewhat ſurpriſed not to find in any one of them a clear Explanation of that moſt curious Paper in N° 219. of the *Philoſophical Tranſactions*, written by the excellent Mathematician Dr. *Halley*; who, not intending to write for Beginners, as himſelf confeſſes, has drawn his Concluſions in a manner, that ſeems to ſtand in need of an Explanation, for the Generality of Readers: And as the maritime People are not the beſt acquainted with mathematical Knowledge, it might have been expected, that ſuch of the Writers on Navigation within the laſt 50 Years, who have undertaken to demonſtrate the ſeveral Parts of their Subject, would have removed the Difficulties in the Doctor's Paper, inſtead of leaving them in the ſame State in which they firſt appeared.

Dr. *Halley*, in this Tract, ſeems to have had two chief Points in View; Firſt, To prove, that *the Diviſions of the Meridian Line in a Mercator's Chart,*

Chart, were analogous to the logarithmic Tangents of the Half-Complements of the Latitudes. And, secondly, *To find a Rule by which the Tables of meridional Parts might be computed from* Briggs's, *or the common logarithmic Tangents.* The former of these the Doctor has clearly and elegantly proved: But he has given rather too few Steps to shew as clearly the Investigation of the latter.

Indeed in many of the Treatises on Fluxions, it is shewn how to investigate a Rule to find the meridional Parts to any Latitude: But, to understand those Methods, requires some Skill in algebraical and fluxionary Computations; neither of which are necessary in this Business, by keeping to the Doctor's Principles, as will be evident from the following Articles; some of which are already well known; yet it was thought convenient to annex them to this Discourse, by

Your most humble Servant,

John Robertson.

Article I. *If the Circumference of a Circle be divided into any Number of equal Parts by as many* Radii, *and a Line be drawn from the Circumference cutting those* Radii, *so that their Parts intercepted between this Line and the Centre be in a continued decreasing geometric Progression; then will that intersecting Line be a Curve, called the proportional Spiral, and will intersect those* Radii *at equal Angles.*

This

This will be evident, by supposing the *Radii* so near to one another, that the intercepted Parts of the Spiral may be taken as right Lines: For then there will be a Series of similar Triangles, each having an equal Angle at the Centre, and the Sides about those Angles proportional.

Art. II. *The same things still supposed, the Parts of the Circumference of the Circle, reckoned from any one Point, may be taken as the Logarithms of the* Ratio's *between the corresponding Rays of the Spiral.*

For those Rays are a Series of Terms in a continued geometric Progression; and the Parts of the Circumference form a Series of Terms in arithmetic Progression. Now the Terms of the arithmetic Series being taken as the Exponents of the corresponding Terms in the geometric Series, there will be the same Relation between each geometric Term and its Correlative, as between Numbers and their Logarithms. And hence the proportional Spiral is also called the logarithmic Spiral.

Art. III. *That proportional Spiral, which intersects its* Radii *at Angles of 45 Degrees, produces Logarithms that are of* Napier's *Kind.*

For, if the Difference between the first and second Terms in the geometric Series was indefinitely small, and the first Division of the Circumference was of the same Magnitude; then may that Part of the Spiral, intercepted between the first and second *Radii*, be taken as the Diagonal of a Square, two of whose Sides are Parts of those *Radii*: Therefore the Spiral which

which cuts its Rays at Angles of 45 Degrees, has a kind of Logarithms belonging to it, so related to their corresponding Numbers, that the smallest Variation between the first and second Terms in the geometric Series, is equal to the Logarithm of the second Term, a Cypher being taken for the Logarithm of the first. But of this kind are the hyperbolical Logarithms, or those first made by their Inventor the Lord *Napier:* Consequently the Logarithms to that Spiral which cuts its Rays at Angles of 45 Degrees, are of the *Napierian* Kind.

Art. IV. *The Rhumb-Lines on the Globe are analogous to the logarithmic Spiral.*

For every oblique Rhumb cuts the Meridian at equal Angles: And it is a Property in stereographic Projections, that the Lines therein intersecting one another, form Angles equal to those which they represent on the Sphere. Therefore a Projection of the Sphere being made on the Plane of the Equator, the Meridians will become the *Radii* of the Equator, and the Rhumbs intersecting them at equal Angles, will become the proportional Spiral.

Hence, the Arcs of the Equator, or the Differences of Longitude reckoned from the same Meridian, are as the Logarithms of those Parts of the corresponding Meridians, intercepted between the Centre and Rhumb-Line.

Art. V. *A Sea Chart being constructed, wherein the Meridians are parallel to one another, and the Lengths of the Degrees of Latitude increase in the same Proportion as the meridional Distances*

ſtances decreaſe on the Globes, will conſtitute a Mercator's Chart; wherein, beſides the Poſitions of Places having the ſame Proportions to one another as on the Globes, the rhumb Lines will be repreſented by right Lines.

For none but right Lines can cut at equal Angles ſeveral parallel right Lines.

Art. VI. *The Diviſions of the meridian Line on a Mercator's Chart, are the ſame as a Table of the Differences of Longitude anſwering to each Minute, or ſmall Difference of Latitude on the rhumb Line making Angles of 45 Degrees with the Meridians.*

For, in ſuch a Chart, the Parallels of Latitude are equal to the Equator, and are at right Angles to the Meridians: And therefore a Rhumb of 45 Degrees cuts the Meridians and Parallels of Latitudes at equal Angles; conſequently between the Interſection of any Meridian and Parallel, and a Rhumb cutting them at 45 Degrees, there muſt be equal Parts of the Meridian and Parallel intercepted: Now, on the Equator, or Parallels of Latitude, are reckoned all the ſucceſſive Differences of Longitudes, and on the Meridians the ſucceſſive meridional Differences of Latitudes, or the Diviſions of the nautical Meridian: Therefore on the Rhumb of 45 Degrees, the ſucceſſive Differences of Longitude are equal to the correſponding Diviſions of the nautical Meridian.

Art. VII. *The Tangents of the Angles which different Rhumbs make with the Meridians, are directly proportional to the Differences of Longi-*

tudes made on those Rhumbs, when the meridional Differences of Latitudes are equal; or, are reciprocally proportional to unequal meridional Differences of Latitudes on those Rhumbs, when the Differences of Longitudes are equal.

For the meridional Difference of Latitude, is to the Difference of Longitude; as *Radius* is to the Tangent of the Angle of the Course, or of the Angle which the Rhumb makes with the Meridian. Therefore, when the meridional Differences of Latitudes are equal, the Differences of Longitudes are as the Tangents of the Courses: But, when the Differences of Longitudes are equal, the meridional Differences of Latitudes are reciprocally as the Tangents of the Courses.

Art. VIII. *The logarithmic Tangents of the Half-Complements of the Latitudes, are analogous to the lengthen'd Degrees in the nautical Meridian Line, in a Mercator's Chart.*

For, in the stereographic Projection of the Sphere on the Plane of the Equator, the Latitudes of Places are projected by the Half-Tangents of the Complements of those Latitudes, which Half-Tangents are the Rays of a proportional Spiral: Now, if a Series of successive Latitudes be taken on any Rhumb, the corresponding Differences of Longitudes will be Logarithms to the Rays of the Spiral, or to the Tangents of the Half-Complements of those Latitudes: Therefore the Differences of Longitudes are as the logarithmic Tangents of the Half-Complements of the Latitudes; But (*Art.* VI.) the lengthened Degrees on the nautical

nautical Meridian are as the Differences of Longitudes on the Rhumb of 45 Degrees; conſequently the logarithmic Tangents of the Half-Complements of Latitudes are as the lengthened Degrees on the nautical Meridian.

Corol. 1. When the Angle between the rhumb Line and the Meridian is equal to 45 Degrees, then the Longitudes of Places on that Rhumb are expreſſed by Logarithms of *Napier*'s Kind; whoſe correſponding Numbers are natural Tangents of the Half-Complements of the Latitudes to Arcs expreſſed in Parts of the *Radius*.

Corol. 2. Hence, to any two Places on a Rhumb of 45 Degrees, the Difference of Longitude, or the meridional Difference of Latitude, is equal to the Difference of the *Napierian* logarithmic Tangents of the Half-Complements of the Latitudes of thoſe Places, eſtimated in Parts of the *Radius*.

Corol. 3. As there may be an indefinite Variety of Rhumbs, and therefore as many different Kinds of Logarithms, conſequently every Species of Logarithms has its peculiar Rhumb, diſtinguiſhable by the Angle it makes with the Meridian: Therefore, among theſe there are two Kinds, whereto the Differences of Longitudes are the Differences of the logarithmic Tangents of the Half-Complements of Latitudes, eſtimated in Minutes of a Degree; one of them belonging to *Napier*'s Form of logarithmic Tangents, and the other to *Briggs*'s, or the common logarithmic Tangents.

Art. IX. *The common logarithmic Tangents are a Table of the Differences of Longitudes, to every*

 Minute

Minute of Latitude, on the rhumb Line making Angles with the Meridians of $51^\circ\ 38'\ 9''$.

For, let z represent the meridional Difference of Latitude between two Places on the Rhumb of 45 Degrees; or its Equal, the Difference between the logarithmic Tangents of the Half-Complements of the Latitudes of those Places, estimated either in Parts of the *Radius*, or in Minutes of a Degree. Then,

As the Circumference in Parts of the *Radius* $= 62831,853$ *&c.*

To the Circumference in Minutes of a Degree $= 21600$.

So is a meridional Difference of Latitude in Parts of the *Radius* $= z$.

To a meridional Difference of Latitude in Minutes of a Degree, $= 0,34377468$ *&c.* $\times z$.

Whose corresponding Rhumb is different from that which z belonged to; and the Angle which this Rhumb makes with the Meridian, will be found by the following Analogy from *Art.* 7.

As the meridional Difference of Latitude on one Rhumb $= 0,34377468$ *&c.* z.

To the meridional Difference of Latitude on a Rhumb of 45 Degrees, $= z$.

So is the natural Tangent of the Rhumb of 45 Degrees, $= 10000$.

To the natural Tangent of the other Rhumb, $= 29088,821$, *&c.*

Which Tangent answers to $71^\circ\ 1'\ 42''$; and this is the Angle that the rhumb Line makes with the Meridians, on which the Differences of the logarithmic Tangents

Tangents of the Half-Complements of the Latitudes, in *Napier's* Form, are the true Differences of Longitudes eftimated in fexagefimal Parts of a Degree.

Now *Napier's* Logarithms being to *Briggs's* as 2,30258 *&c.* is to 1.

Therefore, 2,30258 *&c.* : 1 :: 29088,821 *&c.* : 12633,114 *&c.*; which is the Tangent of 51°. 38′ 9″; and in this Angle are the Meridians interfected by that Rhumb, on which the Differences of *Briggs's* logarithmic Tangents of the Half-Complements of the Latitudes, are the true Differences of Longitudes correfponding to thofe Latitudes.

Art. X. *The Difference between* Briggs's *logarithmic Tangents of the Half-Complements of the Latitudes of any two Places, to the meridional Difference of Latitude in Minutes between thofe Places, is in the conftant Ratio of* 1263,3 *&c.* to 1; or of 1 to 0,0007915704 *&c.*

For *Briggs's* logarithmic Tangents are as the Differences of Longitudes on the Rhumb (*A*) of 51°. 38′ 9″; whofe natural Tangent is 1263,3 *&c.*

The nautical Meridian is a Scale of Longitudes on the Rhumb (*B*) of 45 Degrees, by *Art.* VI. whofe Tangent being equal to the *Radius*, may be expreffed by Unity. And the Differences of Longitude to equal Differences of Latitudes on different Rhumbs, being to each other as the Tangents of the Angles thofe Rhumbs make with the Meridians. Therefore,

As the Tangent of *A* (51° 38′ 9″) = 1,2633, *&c.*
To the Tangent of *B* (45°) = 1,0000;

So

So is the Difference of Longitudes on *A*, or the Difference between the logarithmic Tangents of the Half Co-latitudes of two Places

To the Difference of Longitudes on *B*, or the meridional Difference of Latitudes of those Places.

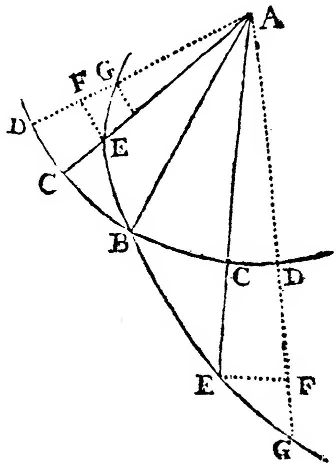
And hence arise the Rules which are given in nautical Works, for finding the meridional Parts by a Table of common logarithmic Tangents.

This curious Discovery of Dr. *Halley*'s, joined to that excellent Thought of his, of delineating the Lines, shewing the Variation of the Compass, on the nautical Chart, are some of the very few useful Additions made to the Art of Navigation within the last 150 Years: For if, beside these, we except the Labours of that ingenious Artist Mr. *Richard Norwood*, who improved the Art by adding to it the Manner of sailing in a Current, and by finding the Measure of a Degree on a great Circle, the Theory of Navigation will be found nearly in the same State in which it was left by that eminent Mathematician Mr. *Edward Wright*; who, about the Year 1600, published the Principles on which the true nautical Art is founded; and shewed, what does not appear to have been known before, how to estimate a Ship's true Place at Sea, as well in Longitude as in Latitude, by the Use of a Table of meridional Parts, first made by himself, and constructed by the constant Addition of the Secants, and which differs almost insensibly from such a Table made on Dr. *Halley*'s Principles, contained in the preceding Articles.

I shall conclude this Discourse with an Article, which, altho' it be somewhat foreign to the preceding Subject, yet, as it was discover'd while I was contem-

plating

templating fome Part thereof, and perhaps is not exhibited in the fame View by others, it may not be improper to annex it in this Place: Which is to demonftrate this common logarithmic Property, that *the Fluxion of a Number divided by that Number, is equal to the Fluxion of the* Napierian *Logarithm of that Number.*



Let BEG be a logarithmic Spiral, cutting its Rays at Angles of 45 Degrees: Then, if AE be taken as a Number, BC will be its *Napierian* or hyperbolic Logarithm.

Alfo, let CD exprefs the Fluxion of the Logarithm BC; and the correfponding Fluxion of the Number AE, will be reprefented by FG, or its Equal FE; as the Angles FEG and FGE are equal.

Now, $AC : CD :: AE : (EF =) FG$.

Therefore $CD = \frac{FG}{AE} \times AB$.

And if AB be taken as the Unit or Term from whence the Numbers begin:

Then $CD = \frac{FG}{AE}$. *Q. e. d.*

VII.